UNIVERSITEIT GENT

FACULTY OF ENGINEERING

Finding Bounds on Ehrhart Quasi-Polynomials

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Outline

1. Introduction
   - What are (Ehrhart) quasi-polynomials?
   - Where do they arise?
   - Why do we need bounds on quasi-polynomials?

2. How do we find bounds?
   - Continuous versus discrete domain extrema of polynomials
   - Converting quasi-polynomials into polynomials

3. Conclusions and Future Work

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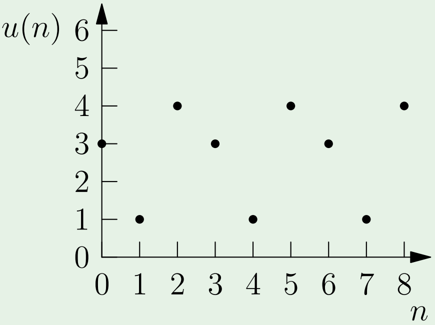
Periodic Numbers

Example



$u(n) = [3, 1, 4]_n$

Periodic Numbers

Definition

Let n be a discrete variable, i.e. $n \in \mathbb{Z}$. A 1-dimensional periodic number is a function that depends periodically on n.

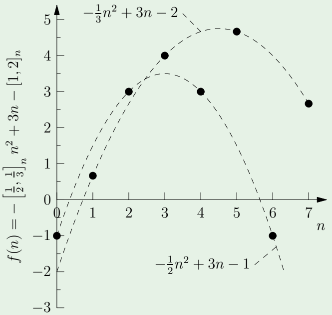
$$u(n) = [u_0, u_1, \ldots, u_{d-1}]_n = \begin{cases} u_0 & \text{if } n \equiv 0 \pmod{d} \\ u_1 & \text{if } n \equiv 1 \pmod{d} \\ \vdots & \\ u_{d-1} & \text{if } n \equiv d-1 \pmod{d} \end{cases}$$

d is called the period.

Quasi-Polynomials

Example

$$f(n) = -\left[\tfrac{1}{2},\tfrac{1}{3}\right]_n n^2 + 3n - [1,2]_n = \begin{cases} -\frac{1}{3}n^2+3n-2 & \text{if } n \equiv 0 \pmod 2 \\ -\frac{1}{2}n^2+3n-1 & \text{if } n \equiv 1 \pmod 2 \end{cases}$$



Quasi-Polynomials

Definition

A polynomial in a variable x is a linear combination of powers of x:

$$f(x) = \sum_{i=0}^{g} c_i x^i$$

Quasi-Polynomials

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A polynomial in a variable x is a linear combination of powers of x:

$$f(x) = \sum_{i=0}^{g} c_i x^i$$

Definition

A quasi-polynomial in a variable x is a polynomial expression with periodic numbers as coefficients:

$$f(n) = \sum_{i=0}^{g} u_i(n) n^i$$

with $u_i(n)$ periodic numbers.

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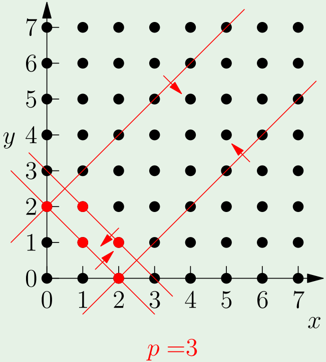
Where do Quasi-Polynomials arise?

Example

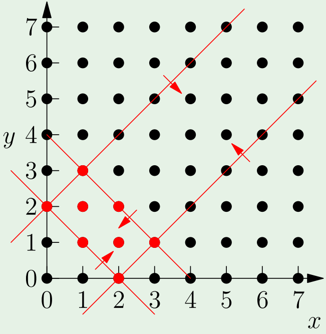


$x + y \leq p$

p	$f(p)$
3	5

Where do Quasi-Polynomials arise?

Example

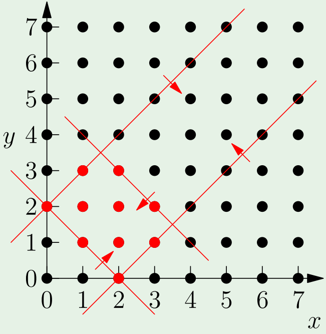


$x + y \leq p$

p	$f(p)$
3	5
4	8

Where do Quasi-Polynomials arise?

Example



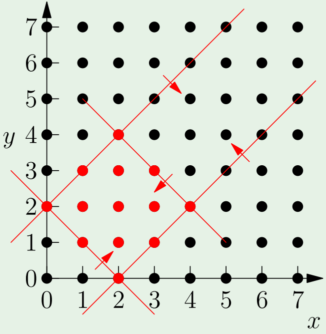
$p = 5$

$x + y \leq p$

p	$f(p)$
3	5
4	8
5	10

Where do Quasi-Polynomials arise?

Example



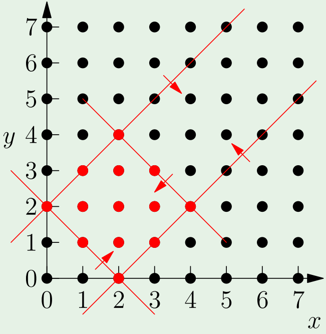
$p = 6$

$x + y \leq p$

p	$f(p)$
3	5
4	8
5	10
6	13

Where do Quasi-Polynomials arise?

Example



$p = 6$

$x + y \leq p$

p	$f(p)$
3	5
4	8
5	10
6	13

$$\frac{5}{2}p + \left[-2, \frac{-5}{2}\right]_p$$

Where do Quasi-Polynomials arise?

- The number of integer points in a parametric polytope P_p of dimension n is expressed as a piecewise a quasi-polynomial of degree n in p (Clauss and Loechner).
- More general polyhedral counting problems:
  Systems of linear inequalities combined with $\vee, \wedge, \neg, \forall$, or $\exists$ (Presburger formulas).
- Many problems in static program analysis can be expressed as polyhedral counting problems.

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Why do we need bounds on quasi-polynomials?

Some problems in static program analysis need bounds on quasi-polynomials.

Example

Number of live elements = quasi-polynomial

$\Downarrow$

Memory usage = maximum over all execution points

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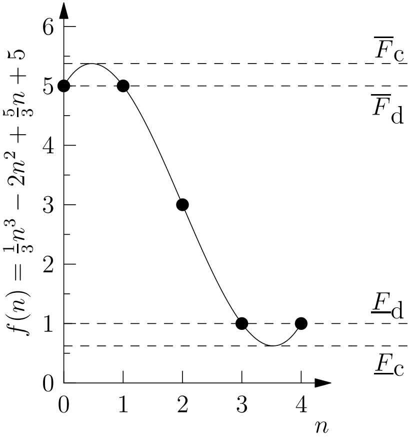
Continuous vs. Discrete domain extrema of polynomials

Discrete domain $\Rightarrow$ evaluate in each point
Not possible for

- parametric domains
- large domains (NP-complete)

Continuous vs. Discrete domain extrema of polynomials



- The relative difference is smaller for
  - larger intervals
  - lower degree
- $\Rightarrow$ Continuous-domain extrema can be used as approximation of discrete-domain extrema.

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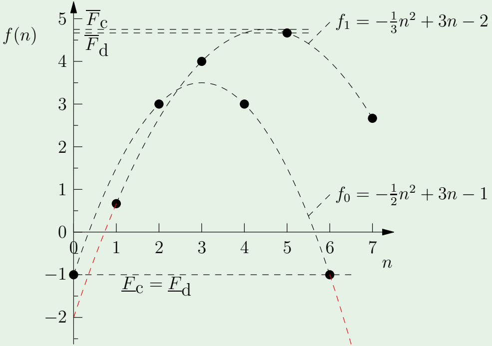
How: Mod Classes

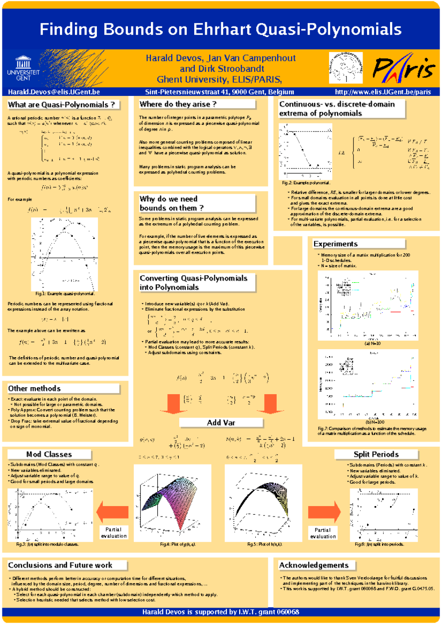
Example



Good for

- small period
- large domains

How: Other Methods



Other methods

- needed for large periods
- offer trade-off between accuracy and computation time
- see poster

Conclusions and Future Work

- Bounds on quasi-polynomials useful for static program analysis
- Different methods fit different situations (period, degree, domain size).

- Outlook
  - A hybrid method should be constructed.
  - Parametric bounds on parameterized quasi-polynomials